Stochastic resonance in a bistable piecewise potential: analytical solution

1996 J. Phys. A: Math. Gen. 29 L447

(http://iopscience.iop.org/0305-4470/29/18/001)

J. Phys. A: Math. Gen. **29** (1996) L447–L452. Printed in the UK

LETTER TO THE EDITOR

Stochastic resonance in a bistable piecewise potential: analytical solution

V Berdichevsky and M Gitterman
Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

Received 20 May 1996, in final form 4 July 1996

Abstract. We consider a bistable piecewise potential subject to both a periodic signal and a random force. The output signal-to-noise ratio (SNR) is calculated for a small-amplitude periodic signal of arbitrary frequency Ω and noise strength D. Our analysis not only recovers the well known non-monotonic dependence of SNR on D at fixed Ω, but also predicts the non-monotonic dependence of SNR on Ω at fixed D. The latter phenomenon does not appear in the commonly assumed adiabatic approximation which applies only at low frequencies.

Stochastic resonance (SR) is a phenomenon found in dynamic nonlinear systems driven by a combination of a periodic signal and a random force. A prototype of a system exhibiting SR is the one-dimensional overdamped process $x(t)$ where the nonlinearity has the form of the symmetric double-well potential:

$$U(x) = -\left(\frac{ax^2}{2} - \frac{bx^4}{4}\right). \tag{1}$$

Manifestation of SR can be divided into two classes. The first related to such characteristics of the motion as $\langle x(t)\rangle$, the autocorrelation function, the power spectrum, or a signal-to-noise ratio. The mathematical form of such SR is often represented in terms of Fourier components of these quantities which behave non-monotonically as a function of the noise amplitude D generally having a maximum at the frequency Ω of the external field. A second group of physically interesting parameters which can show resonant behaviour are characteristic times exemplified by the reciprocal of the switching rate for transitions between the wells defined by equation (1). This can also depend non-monotonically on D.

This sort of non-monotonic behaviour has been found in many physico-chemical and biological systems which are described in numerous articles and summarized in the proceedings of the conference on SR [1] and recent reviews [2–5].

The solution of the nonlinear equation with (1) as a typical potential is quite complicated. The most effective approach is the adiabatic approximation which presumes that the amplitude A and the frequency Ω of the external field as well as the noise strength D are small,

$$A, D, \Omega \ll 1. \tag{2}$$

The most elegant formulae have been obtained for a two-level system [6] while many numerical results are related to continuous systems [2]. There are also some different approaches based on the Floquet-type description [7], perturbation theory [8] and linear response theory [4].

Many attempts have been made to go beyond limitations of equation (2). Some interesting results have been obtained using the next order of the adiabatic approximation

0305-4470/96/180447+06$19.50

[9, 10] which makes it possible to remove the restriction on Ω given in equation (2), and to get analytical results under the conditions

$$A, D \ll 1. \tag{3}$$

The aim of this letter is to obtain exact analytical results for a simple model which exhibits SR provided that

$$A \ll 1. \tag{4}$$

Our model which allows the analytical solution not only reproduces the known non-monotonic dependence of SNR as a function of D for given Ω, but also shows the non-monotonic dependence of SNR on Ω for given D which, to our knowledge, was never seen before. It should be noted, however, that the non-monotonic behaviour in Ω has been found in numerical simulations for the peaks of the distribution of switching times [11].

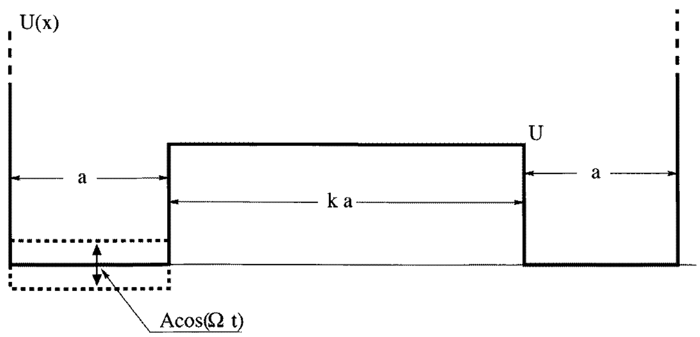

Figure 1. The square-well potential $U(x)$.

We consider a particle moving in the piecewise-symmetric potential shown in figure 1 under the influence of white noise. For this simple potential, we can obtain [12] the exact solution of the full dynamic problem. In order to obtain SR, one has to add a periodic signal $A\cos(\Omega t)$ to the system which we assume to act on the left potential well as is shown in figure 1. The form of the periodic signal is such that for half the period, $2\pi/\Omega$, the left well is deeper than the right well, and *vice versa* during the second half of each period. Such a choice is similar to that of [6] and other papers. A different possibility—a periodic change of barrier height without changing the relative positions of the wells [13]—will probably lead to qualitatively similar results.

The Fokker–Planck equation for the probability density function $P(x, t)$ for the position of a diffusive particle at time t is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left[\frac{1}{T}\frac{\partial U}{\partial x}P + D\frac{\partial P}{\partial x}\right] \equiv -\frac{\partial J}{\partial x} \tag{5}$$

where J is the probability current. For the potential shown in figure 1, $\partial U/\partial x = 0$, and equation (5) reduces to a simple diffusion equation. Our choice for the periodic signal has the advantage that this signal enters only the matching condition (equation (11)) and not the differential equation which will not be the case for the potential described by equation (1).

Substituting $\mathrm{d}U/\mathrm{d}x = 0$ in equation (5) and performing the Fourier transform $\hat{P}(x, \omega) = \int_0^T P(x, t)\exp(-\mathrm{i}\omega t)\,\mathrm{d}t$ with $T \to \infty$ (we assume that $P(x, t) = 0$ at $t < 0$), one can rewrite equation (5) in the following form:

$$P(x,T)\exp(-\mathrm{i}\omega T) - P(x,0) + \mathrm{i}\omega\hat{P}(x,\omega) = D\frac{\mathrm{d}^2\hat{P}(x,\omega)}{\mathrm{d}x^2}. \tag{6}$$

For simplicity we assume that initially a particle is located at the left end of the barrier

$$P(x,0) = \delta(x+a) \tag{7}$$

although it is physically obvious that the qualitative results will not depend on the precise initial position of the particle. The asymptotic distribution function $P(x,T)|_{T\to\infty}$ consists of the stationary distribution $S(x)$ and the small correction term $Af(x,t)$ caused by the oscillations and which is proportional to their amplitude A.

Using these results, one can rewrite equation (6) as

$$[S(x) + Af(x,T)]\exp(-\mathrm{i}\omega T) - \delta(x+a) + \mathrm{i}\omega\hat{P}(x,\omega) = D\frac{\mathrm{d}^2\hat{P}(x,\omega)}{\mathrm{d}x^2}. \tag{8}$$

In each of the three intervals in figure 1, the solution of equation (8) has the form

$$\hat{P}_m(x,\omega) = C_m\exp(rx) + C'_m\exp(-rx) - \frac{S_m}{\mathrm{i}\omega}\exp(-\mathrm{i}\omega T) + Ag_m(x,T)\exp(-\mathrm{i}\omega T) \tag{9}$$

where $m = 1, 2, 3$, $r = \sqrt{\mathrm{i}\omega/D}$, and the asymptotic probability to be in a given interval is proportional to the length of the interval multiplied by $\exp(U/D)$, where U is the potential barrier in this interval, i.e.

$$S_1 = S_3 = K^{-1} \qquad S_2 = K^{-1}\exp\left(\frac{U}{D}\right) \qquad \text{where } K = 2a + ak\exp\left(\frac{U}{D}\right). \tag{10}$$

Explicit forms of the functions $g_m(x,T)$ are not essential in what follows.

Six constants C_m and C'_m, for $m = 1, 2, 3$, can be found from matching at the boundaries. We assume reflecting boundary conditions at the walls, $J(x=0,t) = J[x = a(k+2), t] = 0$. The matching at the boundaries of the barrier is performed by requiring the probability current J to be continuous, while $P(x,t)$ has finite jumps, [14]

$$P(x-0,t)\exp\left(\frac{U(x-0)}{D}\right) = P(x+0,t)\exp\left(\frac{U(x+0)}{D}\right)$$

at the boundaries $x = a$ and $x = a(k+1)$. Let us specify one of the boundary conditions at $x = a$:

$$P_1(a,t)\exp\left(\frac{A}{D}\cos(\Omega t)\right) = P_2(a,t)\exp\left(\frac{U}{D}\right). \tag{11}$$

Performing the Fourier transform of equation (11), taking into account that for small A the Fourier transform of $P(t)\exp[(A/D)\cos(\Omega t)]$ equals $\hat{P}(\omega) + (A/2D)[\hat{P}(\omega+\Omega) + \hat{P}(\omega-\Omega)]$ and substituting $\hat{P}_1$ and $\hat{P}_2$ from equation (9), one can reduce equation (11) to the following form, up to linear terms in A,

$$[\psi_2(r) + Ag_2(a,T)\exp(-\mathrm{i}\omega T)]\exp\left(\frac{-U}{D}\right) = \psi_1(r) + Ag_1(a,T)\exp(-\mathrm{i}\omega T) + \frac{A}{2D}[\psi_1(r') + \psi(r'')] \tag{12}$$

where

$$r' = \frac{\mathrm{i}(\omega-\Omega)}{D} \qquad r'' = \frac{\mathrm{i}(\omega+\Omega)}{D}$$

and

$$\psi_m(r) = C_m(r)\exp(ar) + C'_m \exp(-ar) + \frac{S_m}{\mathrm{i}\omega}\exp(-\mathrm{i}\omega T).$$

Adding to equation (12) five additional matching conditions, one can obtain C_m and C'_m, $m = 1, 2, 3$. A straightforward but tedious calculation yields, for the Fourier transform of x at frequency Ω of the external field,

$$x(\omega)|_{\omega=\Omega} \equiv \int_0^a x\hat{P}_1(x,\omega\to\Omega)\,\mathrm{d}x + \int_a^{a(k+1)} x\hat{P}_2(x,\omega\to\Omega)\,\mathrm{d}x + \int_{a(k+1)}^{a(k+2)} x\hat{P}_3(x,\omega\to\Omega)\,\mathrm{d}x = N + S\delta(\omega-\Omega). \quad (13)$$

We define the SNR in a way similar to [4], namely, $|S/N|^2$. There exist other definitions of the SNR, mostly based on the power spectrum of the signal [2–6]. All such definitions of the SNR will show similar properties.

Thus, we finally arrive at the following result:

$$\mathrm{SNR} = |S/N|^2 \quad (14)$$

where

$$\frac{S}{N} = -\pi r A a^{-1}\,\mathrm{e}^{U/D}[\mathrm{e}^{ra}-1][\mathrm{e}^{ra(1+k)}-1][2\,\mathrm{e}^{U/D}+k]^{-1} \times\{\mathrm{e}^{U/D}[\mathrm{e}^{ra}-1][\mathrm{e}^{rak}-1][\mathrm{e}^{ra}(1-ra)-(1+ra)] -[\mathrm{e}^{2ra}+1][\mathrm{e}^{rak}(1+ra)-(1-ra)]\}^{-1}. \quad (15)$$

Notice that equations (14) and (15), obtained to first order of the amplitude A of the periodic signal, hold for arbitrary values of the dimensionless parameters k, (U/D) and $|ra| = (\Omega a^2/D)^{\frac{1}{2}}$, which determine the width of the potential barrier, its height and the frequency of the external signal, respectively. In particular, our results hold for $(U/D) < 1$ and $\Omega > 1$, when the Kramers approximation does not apply.

The most interesting results from the general expressions (14) and (15) are the limiting forms for small $((\Omega a^2/D) \ll 1)$ and large $((\Omega a^2/D) \gg 1)$ frequencies of the external signal:

$$\mathrm{SNR}|_{(\Omega a^2/D)\ll 1} \approx \frac{\pi^2 A^2 \Omega^2 (1+k)^2 \exp(2U/D)}{4D^2[\exp(U/D)+k]^2(2+k)^2} \quad (16)$$

$$\mathrm{SNR}|_{(\Omega a^2/D)\gg 1} \approx \frac{\pi^2 A^2\,\mathrm{e}^{2U/D}}{a^4(2\,\mathrm{e}^{U/D}+k)^2(\mathrm{e}^{U/D}+1)^2}\left[1+\sqrt{\frac{2D}{\Omega a^2}}\tanh\left(\frac{U}{2D}\right)\right]. \quad (17)$$

As seen from equations (16) and (17), the SNR increases initially as Ω^2, and then reaches the limit value from above, i.e. the SNR has a maximum for some intermediate Ω. In figure 2, we show the SNR defined by equation (14) as a function of frequency Ω for different noise strengths D ranging from 0.5 to 4.

In figure 3, we show the usual manifestation of SR, namely, the non-monotonic dependence of SNR on noise strength D for different frequencies Ω ranging from 0.5 to 10. As expected, the resonance maximum occurs at $D \simeq U$.

The only assumption made in obtaining equation (14) is the smallness of the amplitude A of the periodic signal, while no restrictions have been imposed on the comparative values of three characteristic frequencies arising in our problem, namely the frequency of internal oscillations $\omega_{\mathrm{in}} = D/a^2$, the Kramers frequency [12], $\omega_{\mathrm{kr}} = (D/a^2(k+1))\,\mathrm{e}^{-U/D}$, and the frequency Ω of the external signal.

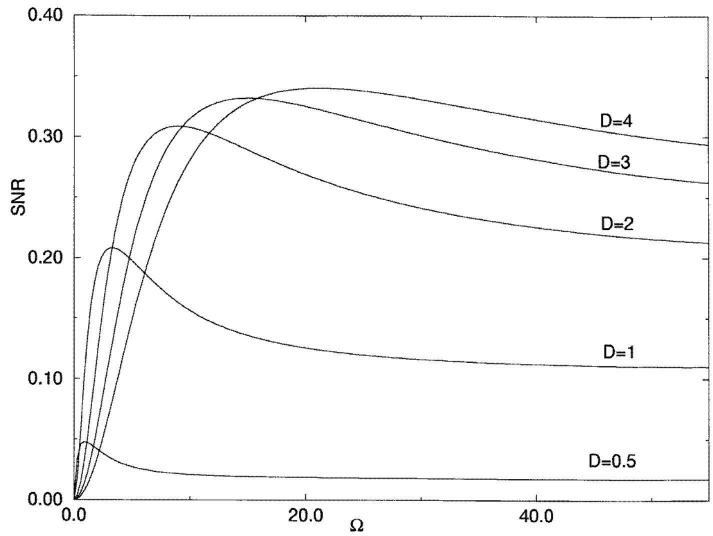

Figure 2. The signal-to-noise ratio (SNR) as a function of the frequency Ω of the external signal for different strengths of noise D for $U = 1.2$ and $k = a = 1$.

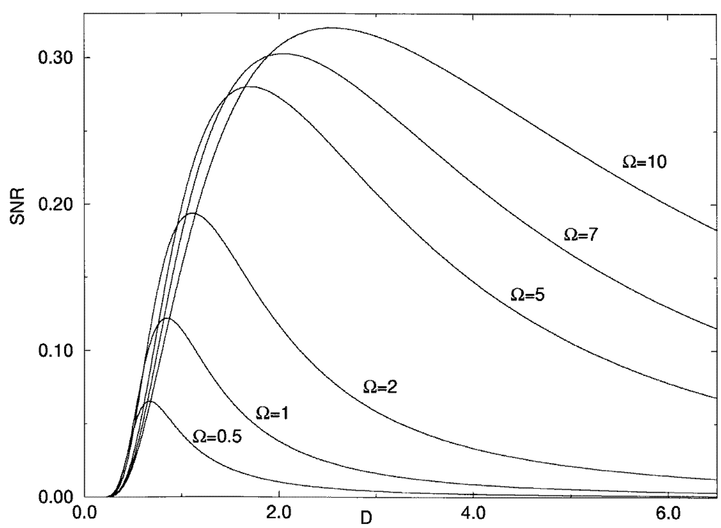

Figure 3. The signal-to-noise ratio (SNR) as a function of the strength of the noise D for different frequencies Ω of the external signal for $U = 1.2$ and $k = a = 1$.

In order to reduce our result to the adiabatic approximation, one has to rewrite equation (14) in the Kramers limit of small noise, $\omega_{\mathrm{kr}} < \omega_{\mathrm{in}}$, which gives

$$\frac{S}{N} = -\frac{\pi A(k+1)\omega_{\mathrm{kr}}}{2a^2\omega_{\mathrm{in}}} \frac{ra[\mathrm{e}^{ra(1+k)} - 1]}{[\mathrm{e}^{rak} - 1][\mathrm{e}^{ra}(1 - ra) - (1 + ra)]}. \tag{18}$$

The adiabatic approximation means that the external frequency Ω is smaller than the characteristic frequencies of the problem which in our case results in $ra \ll 1$. Then, one obtains for the SNR

$$\mathrm{SNR} \approx \frac{\pi^2 A^2 (1+k)^2}{4a^4 k^2}\, \mathrm{e}^{-2U/D}. \tag{19}$$

This limit form is exponentially small in U/D and remains a monotonic function of D in contrast to the two-level problem [6].

In the opposite non-adiabatic case of high external frequency Ω, $\Omega \gg D/a^2$, the limit forms of SNR for narrow ($k \ll 1$) and wide ($k \gg 1$) barriers are

$$\mathrm{SNR} = \begin{cases} \dfrac{\pi^2 A^2 D}{4a^6 \Omega k^2}\, \mathrm{e}^{-2U/D} & \text{for } k \ll 1 \\ \dfrac{\pi^2 A^2}{4a^4}\, \mathrm{e}^{-2U/D} & \text{for } k \gg 1. \end{cases} \tag{20}$$

We hope that our main analytical result—the non-monotonic behaviour of the SNR as a function of the frequency of the external signal—will find use in applications of SR.

References

[1] 1993 *J. Stat. Phys.* **73** no 1/2
[2] Moss F and Weiss G 1994 *Contemporary Problems in Stochastical Physics* (Philadelphia, PA: SIAM) p 205
[3] Jung P 1994 *Phys. Rev.* **234** 175
[4] Dykman M I, Luchinsky G G, Mannela R, McClintock P V E, Stein N D and Stocks N G 1995 *Nuovo Cimento* D **17** 661
[5] Gitterman M and Weiss G 1996 *J. Bifurc. Chaos* to be published
[6] McNamara B and Wiesenfeld K 1989 *Phys. Rev.* A **39** 4854
[7] Jung P and Hänggi P 1991 *Phys. Rev.* A **44** 8032
[8] Presilla C, Marchesoni F and Gammaitoni L 1989 *Phys. Rev.* A **40** 2105
[9] Fox R F and Lu Yan-nan 1993 *Phys. Rev.* E **48** 3390
[10] Hu G, Haken H and Ning C Z 1992 *Phys. Lett.* **172A** 21
[11] Gammaitoni L, Marchesoni L and Santucci S 1995 *Phys. Rev. Lett.* **74** 1052
[12] Berdichevsky V and Gitterman M 1996 *J. Phys. A: Math. Gen.* **29** 1567
[13] Zucher U and Doering C 1993 *Phys. Rev.* E **47** 3862
[14] Risken H 1984 *The Fokker–Planck Equation* (Berlin: Springer)